Packet Loss Characteristics for $M/G/1/N$ Queueing Systems

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Abstract

In this contribution we investigate higher-order loss characteristics for $M/G/1/N$ queueing systems. We focus on the lengths of the loss and non-loss periods as well as on the number of arrivals during these periods. For the analysis, we extend the Markovian state of the queueing system with the time and number of admitted arrivals since the instant where the last loss occurred. By combining transform and matrix techniques, expressions for the various moments of these loss characteristics are found. The approach also yields expressions for the loss probability and the conditional loss probability. Some numerical examples then illustrate our results.

Keywords: Queueing theory, packet loss, renewal theory, Poisson arrivals see time averages.

In wired packet switched networks, packet loss is almost solely caused by buffer overflow in intermediate network nodes. Especially in high-speed networks carrying multimedia or other delay-sensitive traffic, the buffers must be relatively small in order to keep the delay and delay jitter at an acceptable level. Therefore, the user perceived quality of service (QoS) of the traffic flows in such networks is mainly determined by the packet loss process.

An important first-order measure to quantify the loss process is the loss probability or packet loss ratio (PLR), i.e. the fraction of all arriving packets that cannot be accommodated in the buffer. The loss probability has been well studied in queueing literature, see a.o. Cooper (1981) and Takagi (1993). In recent work by Pihlsgard (2005), the PLR is studied in the very general setting of a $GI/G/1/N$ continuous-time queue. Kim and Shroff (2001) further observe that the curve representing the PLR as a function of the buffer capacity N always has a similar shape as the probability of exceeding the level N in an infinite buffer. Hence, various authors have used tail probabilities of the buffer content distribution of an infinite capacity buffer to approximate the PLR (see e.g. Steyaert and Bruneel (1994) and the references therein).

However, the packet loss ratio is not always a sufficient measure to characterise the loss process. For example, for multimedia communications, knowledge of the packet loss ratio alone does not allow one to accurately assess the perceived QoS or quality of experience (QoE). Hadar et al. (2004) compare various visual quality measures of an MPEG-2 encoded video stream subject to both independent and bursty packet losses and note that independent losses have a more severe impact on the visual quality. Verscheure et al. (1998) consider a burst loss model (a Gilbert model) to assess the visual quality of an MPEG-2 stream. Therefore, the frequently used *independence assumption* that losses occur independent from each other with fixed probability (PLR) is unsuited for purposes of performance evaluation.

Hence, authors have studied measures that also capture the higher-order statistics of the loss process of finite buffers. Sheng and Li (1994) investigate the spectrum of the packet loss process whereas the conditional loss probability is investigated by Schulzrinne et al. (1992) and by Takine et al. (1995). Many authors focus on the block loss probability $P(j,n)$ to have j packet losses in a block of n consecutive arrivals. Cidon et al. (1993) calculate the probabilities $P(j,n)$ in a recursive way, both for $M/M/1/N$ and $IPP/M/1/N$ systems. Here, IPP refers to the interrupted Poisson process. The results are compared to the corresponding probabilities as obtained under the independence assumption and it is seen that the latter are a severe underestimation of the exact values. On the other hand, in case a single lost packet causes an entire block of packets to be invalid, then the block loss probability $1-P(0,n)$ is overestimated under the independence assumption. In Gurewitz et al. (2000), the same measure is derived using a purely probabilistic argument based on the classical ballot theorems.

These block loss probabilities are also investigated by different authors to assess the performance of forward error correction (FEC) techniques. FEC may be used to mitigate the effects of packet loss and increase the tolerable packet loss ratio but require that packet loss is well spread in time. With FEC, only k of the n packets in a block carry useful data. The other $n-k$ packets contain redundant data that allow to recover from a limited number of lost packets in the block. Frossard (2001) studies the fraction of invalid blocks after FEC recovery and the mean number of consecutive invalid blocks, assuming both a renewal and a Gilbert model for the loss process. Obviously, adding redundancy with FEC also increases the load of the system, which in turn adds to the number of packets lost. Ait-Hellal et al. (1999) and Altman and Jean-Marie (1998) take this effect into account when studying the efficiency of FEC in case of the $M/M/1/N$ loss process. It is observed that redundancy degrades performance in most cases, but a sufficient amount of FEC can decrease the block loss probability if the load remains below unity. Further, Dube et al. (2003) investigate the performance of FEC in the context of an $M/M/1/N$ queueing system but these authors also take into account additional loss during transmission. More recently, Dán et al. (2006a;b) assess FEC performance to mitigate packet loss of a tagged flow in the presence of Markov modulated Poisson background traffic. In (Dán et al. 2006a) transmission times are either deterministic or exponentially distributed and packets of the tagged flow arrive in accordance with a Poisson process. In (Dán et al. 2006b), transmission times are Erlang distributed and the tagged flow is modelled by means of a Markov modulated Poisson process.

This contribution addresses the evaluation of packet loss characteristics for $M/G/1/N$ queueing systems. In particular, the packet loss characteristics under investigation are the joint distribution of the time and the number of packet arrivals that are *not* lost between consecutive packet losses as well as the lengths of loss periods and the number of arrivals during loss periods. By combining a probability generating functions and Laplace-Stieltjes transform approach with matrix techniques, we obtain expressions for the various moments of these random variables. Although we only focus on the distribution of *loss* and *non-loss* periods, some other measures of the loss process can be obtained from our analysis as

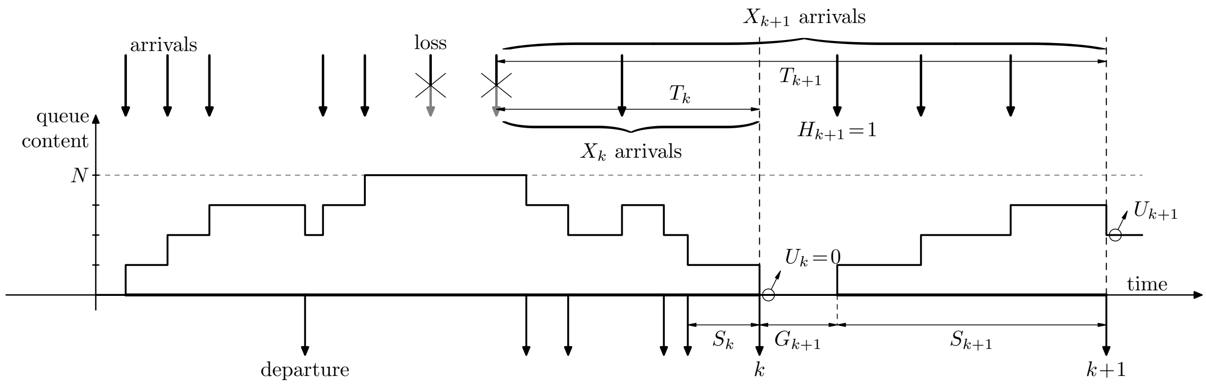

Figure 1: Sample path of the $M/G/1/N$ queueing system where no loss occurs during the transmission of the $(k+1)$st packet.

well. For instance, we give expressions for the loss probability (packet loss ratio), for the conditional loss probability and for the conditional acceptance probability.

This contribution also complements our previous work (Fiems and Bruneel 2005; 2006) on higher-order packet loss characteristics. In (Fiems and Bruneel 2005), the loss characteristics are investigated for discrete-time $M^X/M/1/N$ queueing systems in the context of synchronously operating network nodes. In that paper, the loss characteristics of a selected flow in the presence of background traffic is studied as well. In (Fiems and Bruneel 2006), the loss characteristics of discrete-time and continuous-time Markov-modulated $M/M/1/N$-type queueing systems were investigated. The extension of this previous work to generally distributed transmission times, which is the topic of this paper, is not trivial since we can no longer exploit the lack of memory property of the exponential distribution.

The remainder of this paper is organised as follows. In the next section, we state the details of our model and derive the joint distribution of the number of admitted arrivals and time between consecutive losses. In aid of this analysis, we develop some interesting properties based on renewal theory and the well-known PASTA property, which are given in the Appendix. Section 2 then addresses the characteristics of the loss periods. Finally, we illustrate our results by means of some numerical examples in section 3 and draw conclusions in section 4.

1 Non-loss periods

We investigate packet loss characteristics of an $M/G/1/N$ queueing system. Such a queueing system is completely characterised by the Poisson arrival intensity λ, by the Laplace-Stieltjes transform $S(\zeta)$ of the packet transmission times and by the capacity N of the buffer (including the packet being transmitted). It must be noted that the analysis in this paper is restricted to the case $N > 1$. If $N = 1$, the queueing system has no buffer space and the analysis can be conducted in a similar, although much less involved manner.

1.1 Departure epochs

We first consider the queueing system at packet departure epochs. Let U_k denote the number of packets in the queueing system upon departure of the kth packet and let A_k denote the number of packet arrivals during the kth packet's transmission time. The number of packets in the queueing system upon departure of the $(k+1)$st packet then equals,

(1) $$U_{k+1} = \min((U_k - 1)^+ + A_{k+1}, N - 1)$$

Here $(x)^+$ is the standard shorthand notation for $\max(x, 0)$.

Due to the Poisson nature of the arrival process, the state (in the Markovian sense) of the queueing system on packet departure epochs is completely captured by the random variables U_k. To investigate loss characteristics, we now add some supplementary variables. Let T_k denote the time since the last loss upon departure of the kth packet and let X_k denote the number of packet arrivals since the last loss upon departure of the kth packet.

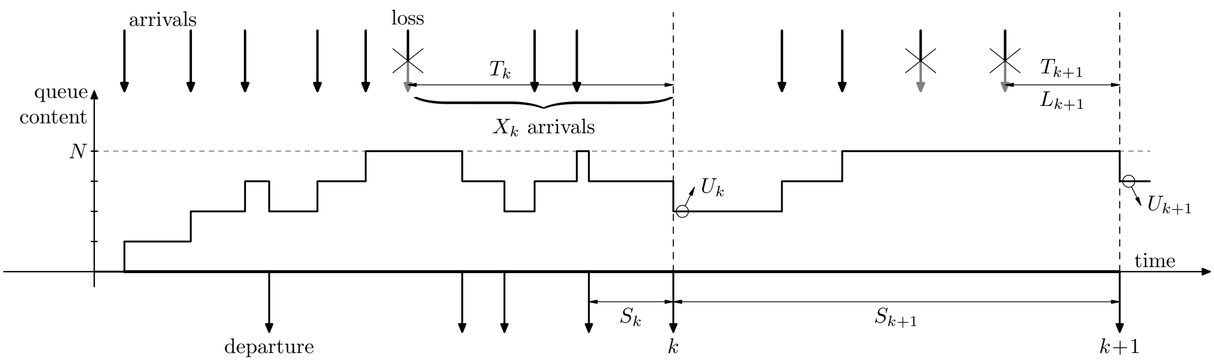

Figure 2: Evolution of the system in case arrivals are lost during the transmission of the $(k+1)$st packet.

For $A_{k+1} \leq N - \max(U_k, 1)$, no packets are lost during the transmission time of the $(k+1)$st packet. Therefore we find,

$$T_{k+1} = T_k + S_{k+1} + G_{k+1}\,, \qquad X_{k+1} = X_k + A_{k+1} + H_{k+1}\,. \tag{2}$$

Here S_k denotes the transmission time of the kth packet. Further, G_k denotes the time between the departure of the $(k-1)$st packet and the time where transmission of the kth packet starts if the $(k-1)$st packet leaves behind an empty buffer and equals 0 if this is not the case. Similarly, H_k equals 1 if the $(k-1)$st packet leaves behind an empty buffer and equals 0 if this is not the case. The equations (2) are illustrated in Fig. 1.

For $A_{k+1} > N - \max(U_k, 1)$, the last $A_{k+1} - (N - \max(U_k, 1))$ arrivals during the $(k+1)$st packet's transmission time are lost as depicted in Fig. 2. We then easily find,

$$T_{k+1} = L_{k+1}\,, \qquad X_{k+1} = 0\,. \tag{3}$$

Here L_k denotes the time between the last arrival during the kth packet's transmission time and the departure epoch of the kth packet if there is at least one arrival during the kth packet's transmission time. Further, we assume that L_k equals the kth transmission time S_k if there are no arrivals during this transmission time ($A_k = 0$) to simplify notation.

Let $P_k(\zeta, z; i)$ denote the transform of T_k and X_k given $U_k = i$,

$$P_k(\zeta, z; i) = \mathrm{E}\left[e^{-\zeta T_k} z^{X_k} 1(U_k = i)\right] \qquad (i = 0, 1, \ldots, N-1)\,.$$

Here $1(x)$ is the indicator function. I.e. $1(x)$ equals 1 if x is true and 0 if this is not the case.

If a departing packet leaves no more than $N-2$ packets behind, there is no loss during this departing packet's transmission time. By conditioning on the number of packets in the buffer upon departure of the kth packet and in view of equations (1) and (2), we find,

$$P_{k+1}(\zeta, z; i) = P_k(\zeta, z; 0)\frac{\lambda z^{i+1}}{\lambda + \zeta}\Theta(\zeta; i) + \sum_{j=1}^{i+1} P_k(\zeta, z; j) z^{i-j+1}\Theta(\zeta; i-j+1) \quad (i = 0, 1, \ldots, N-2)\,, \tag{4}$$

with $\Theta(\zeta; n) = \mathrm{E}\left[e^{-\zeta S_k} 1(A_k = n)\right]$. Notice that the latter Laplace-Stieltjes transform does not depend on k since the consecutive variables (S_k, A_k), $k = 1, 2, \ldots$ constitute a series of independent and identically distributed random variables. Clearly, we have,

$$\Theta(\zeta; n) = \frac{1}{2\pi\imath}\oint_{\mathcal{C}} \Theta(\zeta, z) z^{-n-1} dz\,,$$

with $\Theta(\zeta, z) = \mathrm{E}\left[e^{-\zeta S_k} z^{A_k}\right]$, with $\imath = \sqrt{-1}$ and with $\mathcal{C}$ a contour within the unit disk around 0. The reader is referred to Property 2 in the Appendix for an expression for $\Theta(\zeta, z)$.

If a departing packet leaves behind $N-1$ packets, there may be loss during this departing packet's transmission time. By conditioning on the number of packets in the buffer upon departure of the kth packet and on the number of arrivals during the transmission time of the $(k+1)$st packet and by plugging in equations (1) to (3) we find,

$$P_{k+1}(\zeta,z;N-1) = P_k(\zeta,z;0)\frac{\lambda z^N}{\lambda+\zeta}\Theta(\zeta,N-1) + P_k(0,1;0)\sum_{n=N}^{\infty}\Psi(\zeta;n) + \sum_{j=1}^{N-1} P_k(\zeta,z;j)z^{N-j}\Theta(\zeta;N-j) + \sum_{j=1}^{N-1} P_k(0,1;j)\sum_{n=N-j+1}^{\infty}\Psi(\zeta;n)\,, \tag{5}$$

with,

$$\Psi(\zeta;n) = \mathrm{E}\left[e^{-\zeta L_k} 1(A_k = n)\right] = \frac{1}{2\pi\imath}\oint_{\mathcal{C}} \Psi(\zeta,z) z^{-n-1} dz\,,$$

and with $\Psi(\zeta,z) = \mathrm{E}\left[e^{-\zeta L_k} z^{A_k}\right]$. Again, notice that the transform $\Psi(\zeta,z)$ does not depend on k since the consecutive variables (S_k, L_k, A_k), $k = 1,2,\ldots$ constitute a series of independent and identically distributed random variables. The reader is referred to Property 3 in the Appendix for an expression for $\Psi(\zeta,z)$.

Let $P(\zeta,z;i) = \lim_{k\to\infty} P_k(\zeta,z;i)$, $i = 0,\ldots,N-1$ and let $P(\zeta,z)$ denote the $1\times N$ row vector with elements $P(\zeta,z;i)$. The latter then satisfies the following equation,

$$P(\zeta,z) = P(\zeta,z)\mathcal{A}(\zeta,z) + P(0,1)\mathcal{B}(\zeta)\,, \tag{6}$$

where both $\mathcal{A}(\zeta,z)$ and $\mathcal{B}(\zeta)$ are matrices of size $N\times N$. From equations (4) and (5) it is seen that the elements of $\mathcal{A}(\zeta,z)$ are given by,

$$a_{j+1,i+1}(\zeta,z) = \begin{cases} \dfrac{\lambda z^{i+1}}{\lambda+\zeta}\Theta(\zeta;i) & \text{if } j = 0, \\ z^{i-j+1}\Theta(\zeta;i-j+1) & \text{if } 0 < j \le i+1, \\ 0 & \text{otherwise;} \end{cases}$$

for $i,j = 0,\ldots,N-1$ whereas the matrix $\mathcal{B}(\zeta)$ has elements,

$$b_{j+1,i+1}(\zeta) = \begin{cases} \Psi(\zeta,1) - \displaystyle\sum_{n=0}^{\min(N-1,N-j)} \Psi(\zeta;n) & \text{if } i = N-1, \\ 0 & \text{otherwise;} \end{cases}$$

for $i,j = 0,\ldots,N-1$. Notice that the matrix $\mathcal{A}$ has an upper Hessenberg structure whereas the matrix $\mathcal{B}$ has only non-zero elements in its rightmost column.

Plugging in $\zeta = 0$ and $z = 1$ into equation (6) and taking into account the normalisation condition $P(0,1)e^T = 1$ allows us to determine the unknown vector $P(0,1)$. In fact $P(0,1)$ is the normalised Perron-Frobenius eigenvector of the upper Hessenberg matrix $\mathcal{A}(0,1)+\mathcal{B}(0)$. Here, e^T is a column vector with all elements equal to 1. Given $P(0,1)$, the unknown vector $P(\zeta,z)$ follows from,

$$P(\zeta,z) = P(0,1)\mathcal{B}(\zeta)(\mathcal{I}-\mathcal{A}(\zeta,z))^{-1}\,, \tag{7}$$

where $\mathcal{I}$ denotes the $N\times N$ unit matrix.

For large N, it may become infeasible to compute the matrix inversion in the former expression symbolically. By means of the moment generating property of Laplace-Stieltjes transforms and probability generating functions, we may however obtain similar expressions for the moments which do not require symbolic matrix inversion. In particular, let $\mathrm{E}[T]$ and $\mathrm{E}[X]$ denote the vectors with elements $\mathrm{E}[T_k 1(U_k = i)]$ and $\mathrm{E}[X_k 1(U_k = i)]$ respectively (for $i = 0,\ldots,N-1$ and with k a random packet in steady state). We easily find,

$$\mathrm{E}[T] = -P(0,1)(\mathcal{A}_\zeta(0,1)+\mathcal{B}_\zeta(0))(\mathcal{I}-\mathcal{A}(0,1))^{-1}\,, \qquad \mathrm{E}[X] = P(0,1)\mathcal{A}_z(0,1)(\mathcal{I}-\mathcal{A}(0,1))^{-1}\,.$$

Here $\mathcal{A}_\zeta(\zeta,z)$ and $\mathcal{A}_z(\zeta,z)$ denote the partial derivatives of $\mathcal{A}(\zeta,z)$ with respect to ζ and z respectively and $\mathcal{B}_\zeta(\zeta)$ denotes the derivative of $\mathcal{B}(\zeta)$.

1.2 Transmission epochs

Let $Q(\zeta, z; i)$ denote the transform of the time $\hat{T}$ and the number of arrivals $\hat{X}$ since the last loss for a given buffer content $\hat{U} = i$ when transmission of a random (tagged) packet starts,

$$Q(\zeta, z; i) = \mathrm{E}\left[e^{-\zeta \hat{T}} z^{\hat{X}} 1(\hat{U} = i)\right] \qquad (i = 1, \ldots, N-1).$$

Notice that the buffer content at the start of a packet's transmission can never exceed $N-1$. Further, let T, X and U denote the time since the last loss, the number of arrivals since the last loss and the queue content upon departure of the preceding packet. Let G denote the time between the departure of the preceding packet and the arrival of the tagged packet if the preceding packet leaves behind an empty buffer and let $G = 0$ if this is not the case. Similarly, let H equal 1 if the preceding packet leaves behind an empty buffer and 0 if this is not the case. The triple $(\hat{T}, \hat{X}, \hat{U})$ then relates to the triple (T, X, U) as follows,

$$\hat{U} = U + H, \quad \hat{T} = T + G, \quad \hat{X} = X + H.$$

Since the arrival process of the queueing system under consideration is a Poisson process, one easily observes that the random variable G given $U > 0$ is exponentially distributed with mean $1/\lambda$. By conditioning on the buffer content upon departure of the preceding packet and in view of the preceding equation, we find,

$$\begin{aligned} &Q(\zeta, z; 1) = P(\zeta, z; 0)\frac{\lambda z}{\lambda + \zeta} + P(\zeta, z; 1), \\ &Q(\zeta, z; i) = P(\zeta, z; i) \qquad (i = 2, \ldots, N-1). \end{aligned} \tag{8}$$

Here we implicitly used the fact that the departure epoch preceding the epoch where transmission of a random packet starts is a random departure epoch. The relation (**??**) between the distributions at departure epochs on the one hand and at transmission starts on the other can also be written in the following matrix form,

$$Q(\zeta, z) = P(\zeta, z)\mathcal{D}(\zeta, z), \tag{9}$$

where $Q(\zeta, z)$ is the $1 \times (N-1)$ row vector with the functions $Q(\zeta, z; i)$ as its elements and where the $N \times (N-1)$ matrix $\mathcal{D}(\zeta, z)$ has elements

$$d_{i+1,j} = \begin{cases} \dfrac{\lambda z}{\lambda + \zeta} & \text{if } i = 0, j = 1, \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 0, \ldots, N-1$ and $j = 1, \ldots, N-1$. From (9), the row vectors $\mathrm{E}[\hat{T}]$ and $\mathrm{E}[\hat{X}]$ with elements $\mathrm{E}[\hat{T}\ 1(\hat{U} = i)]$ and $\mathrm{E}[\hat{X}\ 1(\hat{U} = i)]$, $i = 1, \ldots, N-1$ respectively are then found as

$$\mathrm{E}[\hat{T}] = \mathrm{E}[T]\,\mathcal{D}(0,1) - P(0,1)\mathcal{D}_\zeta(0,1), \qquad \mathrm{E}[\hat{X}] = \mathrm{E}[X]\,\mathcal{D}(0,1) + P(0,1)\mathcal{D}_z(0,1),$$

with partial derivatives indicated by the indices ζ and z as before. Note that in $\mathcal{D}_\zeta(0,1)$ and $\mathcal{D}_z(0,1)$, only the element on the first row and the first column is non-zero, with values $-1/\lambda$ and 1 respectively.

1.3 Loss epochs

Now the distribution of the state at the start of a random packet's transmission can be related to the corresponding distribution at a random loss epoch in the following way. Consider a random (tagged) arrival that arrives when a packet is being transmitted. Let $R(\zeta, z)$ denote the transform of the time $\tilde{T}$ and the number of packet arrivals $\tilde{X}$ since the last loss, given that there are $\tilde{U} = N$ packets in the buffer upon arrival of the tagged packet,

$$R(\zeta, z) = \mathrm{E}\left[e^{-\zeta \tilde{T}} z^{\tilde{X}} 1(\tilde{U} = N)\right].$$

Since a packet that arrives in a fully occupied buffer is lost, one easily verifies that

$$V(\zeta, z) = \mathrm{E}\left[e^{-\zeta \tilde{T}} z^{\tilde{X}}\right] = \frac{R(\zeta, z)}{R(0,1)}, \tag{10}$$

is the transform of the time and the number of packet arrivals between consecutive losses, which is our main result. To obtain $R(\zeta, z)$ and hence also $V(\zeta, z)$, we proceed as follows. Let $\hat{U}$ denote the buffer content at the start of the transmission of the packet being transmitted at the tagged packet's arrival epoch. Further, let A_B denote the number of arrivals during the ongoing transmission but before the tagged packet and let T_B denote the time between the start of the ongoing transmission and the arrival epoch of the tagged packet. Then, the tagged arrival finds N packets upon arrival in the buffer whenever $\hat{U} + A_B \geq N$. For $\hat{U} + A_B = N$, the tagged arrival is the first packet that is lost during the ongoing transmission and therefore,

$$\tilde{T} = \hat{T} + T_B , \quad \tilde{X} = \hat{X} + A_B . \tag{11}$$

Recall that $\hat{T}$ ($\hat{X}$) denotes the time (the number of arrivals) since the last loss when transmission of a random packet starts. When $\hat{U} + A_B > N$, the arrival before the tagged arrival is also lost. In that case, there are no packet arrivals since the last loss upon arrival of the tagged packet and the time since the last lost equals the interarrival time between the tagged packet and the preceding packet. As such, we find,

$$\tilde{T} = \tilde{L} , \quad \tilde{X} = 0 . \tag{12}$$

Here $\tilde{L}$ denotes the time between the tagged arrival and the preceding arrival.

Conditioning on the buffer content $\hat{U}$ at the start of the transmission and on the number of packet arrivals during transmission but before the tagged arrival and plugging in equations (11) and (12) then leads to,

$$R(\zeta, z) = \sum_{i=1}^{N-1} Q(\zeta, z; i)\Omega(\zeta; N-i) z^{N-i} + \sum_{i=1}^{N-1} Q(0, 1; i) \left(\Gamma(\zeta, 1) - \sum_{j=0}^{N-i} \Gamma(\zeta; j) \right) ,$$

with $\Omega(\zeta; j) = \mathrm{E}\left[e^{-\zeta T_B} 1(A_B = j)\right]$, with $\Gamma(\zeta, z) = \mathrm{E}\left[e^{-\zeta \tilde{L}} z^{A_B}\right]$ and with,

$$\Gamma(\zeta; j) = \mathrm{E}\left[e^{-\zeta \tilde{L}} 1(A_B = j)\right] = \frac{1}{2\pi\imath} \oint_{\mathcal{C}} \Gamma(\zeta, z) z^{-j-1} dz .$$

Expressions for the unknown functions $\Omega(\zeta; j)$ and $\Gamma(\zeta, z)$ can be retrieved from Properties 4 and 5 in the Appendix.

The mean length $\mathrm{E}[\tilde{T}]$ of a non-loss period and the mean number of admitted arrivals in such a period now easily follow from (10),

$$\mathrm{E}[\tilde{T}] = -V_\zeta(0, 1) = -\frac{R_\zeta(0, 1)}{R(0, 1)} , \qquad \mathrm{E}[\tilde{X}] = V_z(0, 1) = \frac{R_z(0, 1)}{R(0, 1)} .$$

The packet loss ratio, i.e. the fraction of lost packets out of all arriving packets is then simply,

$$\mathrm{PLR} = \frac{1}{1 + \mathrm{E}[\tilde{X}]} ,$$

since every lost packet is followed by an average of $\mathrm{E}[\tilde{X}]$ admitted arrivals.

Another interesting measure that has been used in the past — see a.o. Schulzrinne et al. (1992), Takine et al. (1995) — to characterise correlation in the loss process is the conditional loss probability (CLP), i.e. the probability that a loss is immediately followed by another loss. Again, from (10) we find,

$$\mathrm{CLP} = V(0, 0) = \frac{R(0, 0)}{R(0, 1)} .$$

Similarly, we define the conditional acceptance probability (CAP) as the probability that the buffer can accommodate a packet, given that it could also accommodate the preceding packet. Since a packet is either lost or accepted, we find,

$$\mathrm{CAP} = 1 - \mathrm{PLR} \frac{1 - \mathrm{CLP}}{1 - \mathrm{PLR}} .$$

1.4 Complexity

We now evaluate the complexity of finding the various performance measures in terms of the buffer capacity N. Clearly, for each performance measure one needs to find the normalised Perron-Frobenius eigenvector $P(0,1)$ first, which has complexity $O(N^2)$ since the matrix $(\mathcal{A}(0,1)+\mathcal{B}(0))$ has an upper Hessenberg structure. Differentiation of (7) and evaluation in $\zeta=0$ and $z=1$ allows to find the various derivatives of $P(\zeta,z)$. One thus finds these derivatives by means of some vector matrix multiplications and by solving for the unknown vector. Here the complexity of these operations is also $O(N^2)$ since all involved matrices have an upper Hessenberg structure. Similarly one finds $R(\zeta,z)$ and its derivatives evaluated in $\zeta=0$ and $z=1$ given $P(\zeta,z)$ (and its derivatives). Again the involved matrices have an upper Hessenberg structure and hence the complexity is $O(N^2)$.

So far we have not taken into account the complexity of finding the elements of the various matrices. Clearly, to construct these matrices one needs to find one or more probability series $\{\Pr[Y=n], n=0,\ldots,N\}$ and moment series $\{\mathrm{E}\left[Z\,1(Y=n)\right], n=0,\ldots,N\}$, given the corresponding transforms $\sum_n \Pr[Y=n]z^n$ and $\sum_n \mathrm{E}\left[Z\,1(Y=n)\right]z^n$ respectively (see the Appendix). Only in particular cases, explicit expressions for these series can be found. However, one may rely on the inverse fast Fourier transform (FFT) to find these series. The complexity of the inverse FFT is $O(N\log N)$ (see a.o. Abate and Whitt (1992)). Overall one thus finds that complexity to obtain the various performance measures is $O(N^2)$.

2 Loss periods

Let a loss period be defined as the amount of time between the first and the last arrival of a series of consecutive arrivals that are lost. Notice that the loss period has length zero if there is only one loss during this period. We now retrieve the joint transform $Y(\zeta,z)$ of the length $\overline{T}$ of a random loss period and the number of losses $\overline{X}$ during such a period,

$$Y(\zeta,z)=\mathrm{E}\left[e^{-\zeta\overline{T}}z^{\overline{X}}\right].$$

As opposed to non-loss periods, loss periods start and end during the same transmission. After all, the buffer can accommodate the first arrival after a departure from the queueing system. Therefore, consider a random packet's transmission time. As before, let S denote the transmission time of this packet and let A denote the number of arrivals during this transmission. Further, let $\hat{U}$ denote the buffer content when transmission of this packet starts. Clearly, there is a one-to-one correspondence between loss periods and transmission times where $\hat{U}+A>N$. That is, we have,

$$Y(\zeta,z)=\mathrm{E}\left[e^{-\zeta\overline{L}_{N-\hat{U}+1}}z^{\hat{U}+A-N}|\hat{U}+A>N\right],$$

where $\overline{L}_i$ denotes the time between the ith packet arrival and the last packet arrival during the transmission time S if there are at least $i+1$ arrivals during this transmission. $\overline{L}_i$ equals 0 if this is not the case. Conditioning on the buffer content at the start of the packet transmission time further yields,

$$Y(\zeta,z)=\frac{\sum_{i=1}^{N-1}\Pr[\hat{U}=i]\,\mathrm{E}\left[e^{-\zeta\overline{L}_{N-i+1}}z^{A}1(A>N-i)\right]z^{i-N}}{\sum_{i=1}^{N-1}\Pr[\hat{U}=i]\Pr[A>N-i]}.$$

Notice that the probability $\Pr[\hat{U}=i]$ can be obtained as the ith element of the vector $Q(0,1)$ as determined in the previous section.

Finally, conditioning on the number of arrivals during the transmission time yields,

$$Y(\zeta,z)=\frac{\sum_{i=1}^{N-1}\Pr[\hat{U}=i]\left[\Xi(\zeta,z;N-i+1)-\sum_{n=0}^{N-i}\Xi(\zeta;n,N-i+1)z^{n+i-N}\right]}{\sum_{i=1}^{N-1}\Pr[\hat{U}=i]\left[1-\sum_{n=0}^{N-i}\Xi(0,1;n)\right]},$$

with $\Xi(\zeta,z;i)=\mathrm{E}\left[e^{-\zeta\overline{L}_i}z^{A}\right]$ and $\Xi(\zeta;k,i)=\mathrm{E}\left[e^{-\zeta\overline{L}_i}1(A=k)\right]=\frac{1}{2\pi\imath}\oint_C\Xi(\zeta,z;i)z^{-k-1}dz$. Again, an expression for the unknown transform $\Xi(\zeta,z;i)$ can be retrieved by means of renewal theory and the PASTA property. The reader is referred to Property 6 in the Appendix.

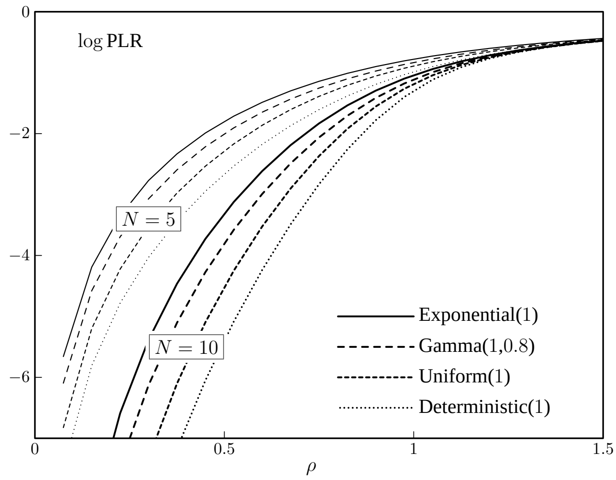

Figure 3: Packet loss ratio vs. arrival load ρ for different transmission time distributions and for $N = 5, 10$.

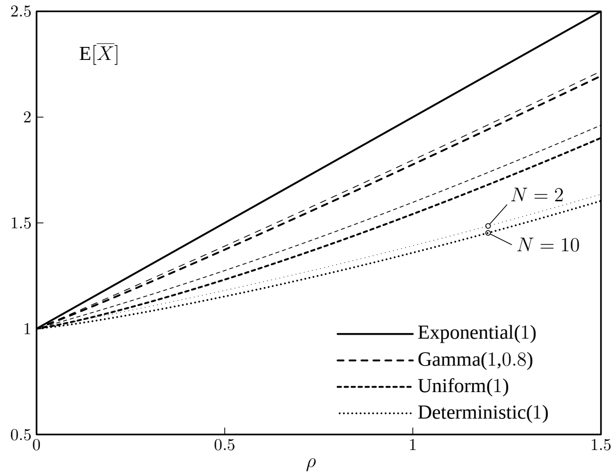

Figure 4: Mean number of losses in a loss period vs. arrival load ρ for different transmission time distributions and for $N = 2, 10$.

3 A numerical example

We now illustrate our approach by means of some numerical examples. In particular, we consider four different distributions for the packet transmission times, all with expected value μ and with respective Laplace-Stieltjes transforms

$$(13) \qquad S_1(\zeta) = (1+\zeta\mu)^{-1}\ , \quad S_2(\zeta) = \left(1+\zeta\frac{\sigma^2}{\mu}\right)^{-\frac{\mu^2}{\sigma^2}}\ , \quad S_3(\zeta) = \frac{1-e^{-2\mu\zeta}}{2\mu\zeta}\ , \quad S_4(\zeta) = e^{-\zeta\mu}\ .$$

These transforms represent transmission times which are (1) exponentially distributed, (2) gamma distributed, (3) uniformly distributed between 0 and 2μ and (4) deterministically equal to μ. Recall that the gamma distribution is completely specified by the mean μ and its standard deviation σ and that it reduces to an exponential distribution for $\mu = \sigma$. In the following examples, we have taken $\mu = 1$ such that the load of the system is always given by $\rho = \lambda\mu = \lambda$. This choice implies that the standard deviation of the chosen transmission time distributions is $\sigma_1 = 1$, $\sigma_3 = 0.577$ and $\sigma_4 = 0$, whereas the standard deviation of the gamma distribution is chosen independently as $\sigma_2 = 0.8$. Note that for these parameters, the distributions in (13) have decreasing variances, i.e. $\sigma_1 > \sigma_2 > \sigma_3 > \sigma_4$. The curves in Figs. 3–10 were obtained by implementing the equations in the previous sections numerically.

In Fig. 3, the packet loss ratio (PLR) is depicted on a logarithmic scale as a function of the arrival load $\rho = \lambda\mu$ for buffer capacity $N = 5, 10$ and for the different transmission time distributions in (13). As

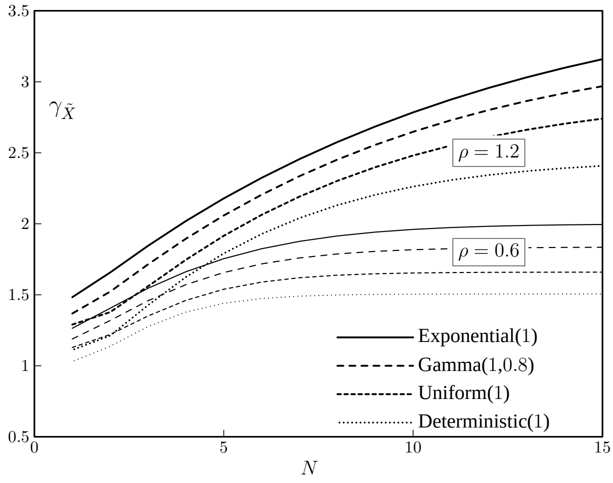

Figure 5: Coefficient of variation $\gamma_{\tilde{X}}$ of the number of arrivals between losses vs. the buffer capacity N, for different transmission time distributions and with $\rho = 0.6, 1.2$.

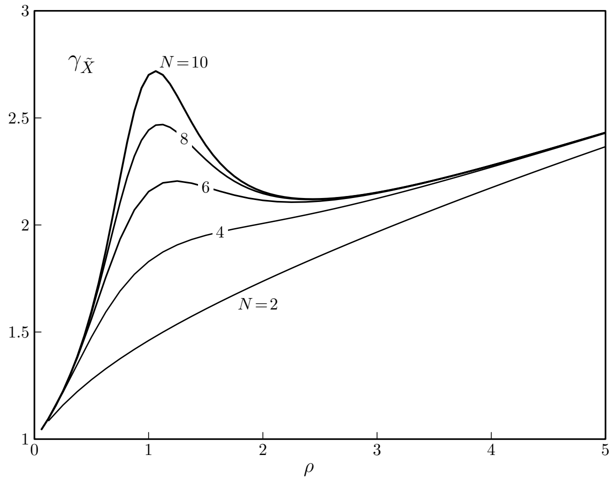

Figure 6: Coefficient of variation $\gamma_{\tilde{X}}$ of the number of arrivals between losses vs. the load ρ for $N = 2, 4, 6, 8, 10$ and for gamma distributed transmission times.

expected, the packet loss ratio increases when the load increases. Further, for $\rho \to \infty$, all curves converge to PLR $= 1$. For high load, most of the packets arrive in a full buffer — whatever the capacity — and are lost. Therefore, the packet loss ratio no longer depends on the buffer capacity. Fig. 4 shows the mean number of losses $\mathrm{E}\left[\overline{X}\right]$ in a loss period as a function of the arrival load for the same transmission time distributions and for buffer capacities $N = 2, 10$. We note that for higher values of N, the curves no longer change, which indicates an insensitivity of $\mathrm{E}\left[\overline{X}\right]$ of the buffer capacity if N is not small. If the transmission times are exponentially distributed (i.e. for a $M/M/1/N$ queueing system), the value of $\mathrm{E}\left[\overline{X}\right]$ is the same for *all* N. The same insensitivity result can be observed for the mean length of a loss period, as shown clearly in Fig. 8 where $\mathrm{E}\left[\overline{T}\right]$ is plotted as a function of N.

The PLR however, is a first-order characteristic of the packet loss process and is often insufficient for performance evaluation purposes. For e.g. the evaluation of the quality of video traffic, the *burstiness* of the losses and of the non-loss packets is of importance as well. Suitable second-order statistics of $\tilde{X}$ and $\tilde{T}$ reveal such information and can be obtained from our analysis in the previous section. For instance, in Fig. 5, the coefficient of variation $\gamma_{\tilde{X}}$ of the number of arrivals between losses is depicted versus the buffer capacity N. Recall that the coefficient of variation of a random variable X with mean μ_X and variance σ_X^2 equals $\gamma_X = \sigma_X/\mu_X$.

Again, we show curves for each of the transmission time distributions in (13) with $\mu = 1$, and for loads $\rho = 0.6$ and $\rho = 1.2$ (overload) respectively. The coefficient of variation increases when the variance

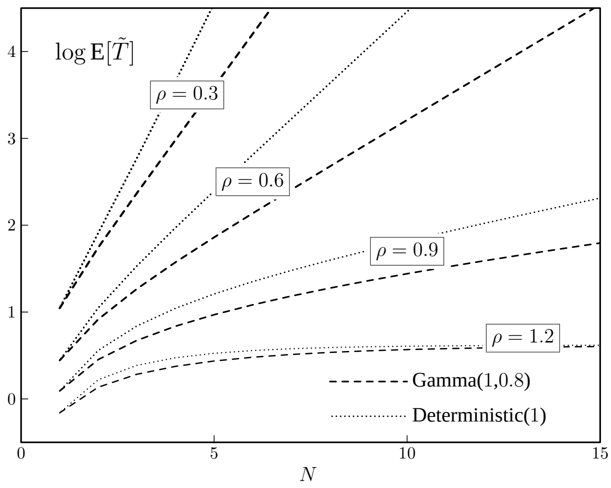

Figure 7: Mean time between losses $\mathrm{E}[\tilde{T}]$ vs. the buffer capacity N for gamma and deterministic transmission times and for various values of the load ρ.

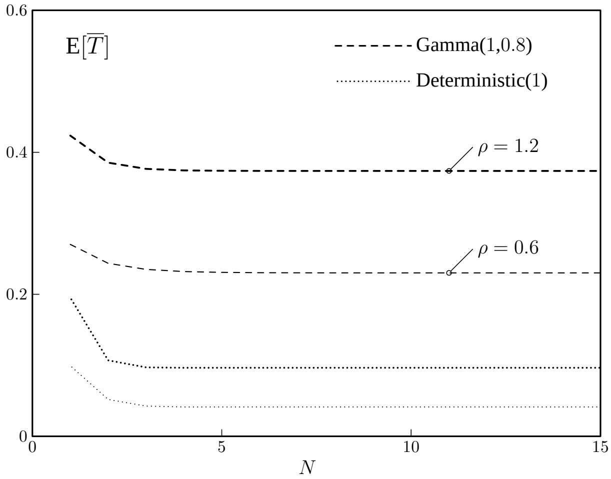

Figure 8: Mean length of a loss period $\mathrm{E}[\overline{T}]$ vs. the buffer capacity N for gamma and deterministic transmission times and for $\rho = 0.6, 1.2$.

of the transmission times increases, as expected. For some values of ρ, $\gamma_{\tilde{X}}$ seems to be rather insensitive of N (e.g. for $\rho = 0.6$ and for sufficiently large N) whereas for other values of ρ, $\gamma_{\tilde{X}}$ keeps increasing with N. We investigate this further in Fig. 6 where $\gamma_{\tilde{X}}$ is shown (for gamma distributed transmission times only) as a function of the arrival load ρ. For ρ approaching 0, we observe that $\gamma_{\tilde{X}}$ converges to 1 for all N. This implies that the distribution of $\tilde{X}$ converges to a geometrical distribution. Indeed, almost all consecutive packets arrive in an empty buffer and therefore the distribution of the number of arrivals after these packets until the next loss is identical. Further, for $\rho \to \infty$ all curves converge according to $\sqrt{\rho - 1}$, independent of N. This is because for high load $\tilde{X}$ is approximately Bernoulli distributed with mean $1/\rho$. Finally, a peak in the variance occurs around the start of overload, which increases with N in accordance with the peak that can be observed for the coefficient of variation of the steady state buffer content.

In Fig. 7, we show a logarithmic plot of the mean time between losses versus the buffer capacity N for different values of the arrival load ρ. Only the results for gamma distributed and deterministic transmission times are shown. Clearly, increasing the buffer capacity implies that there is less packet loss and therefore the mean time between losses is longer. Also, higher load implies more loss and the time between losses is shorter. For $\rho < 1$, the mean time between losses seems to increases exponentially with N (for high N), whereas $\mathrm{E}[\tilde{T}]$ saturates to a fixed value if $\rho > 1$.

Finally, in Fig. 9 and Fig. 10 we show the logarithmic conditional loss and acceptance probability

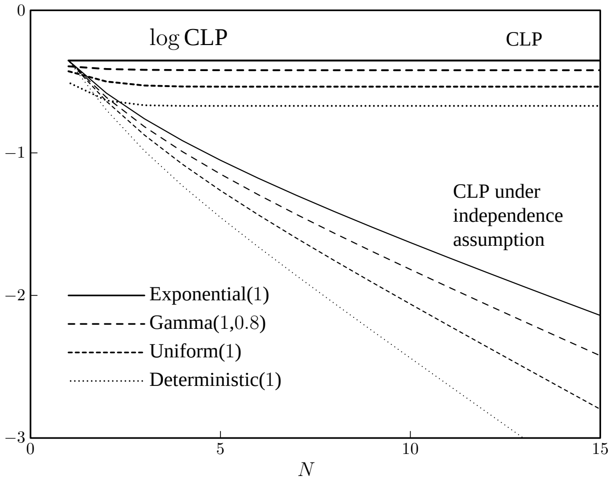

Figure 9: Conditional Loss Probability vs. the buffer capacity N for $\rho = 0.8$, compared with the independence assumption.

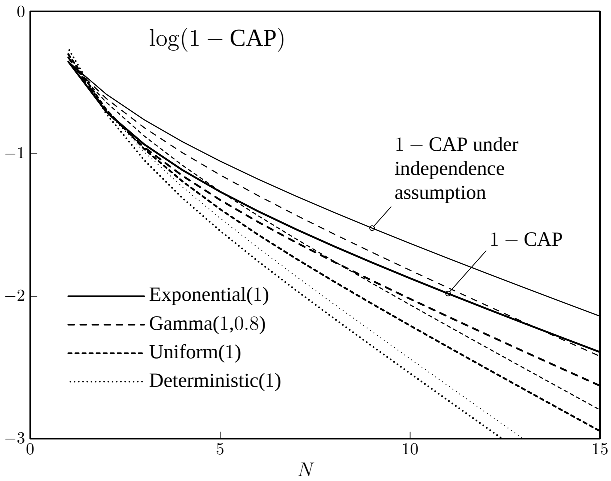

Figure 10: $1 - \text{CAP}$ vs. the buffer capacity N for $\rho = 0.8$, compared with the independence assumption.

respectively. The curves were calculated for a load of 0.8. A high CLP indicates a high degree of 'clustering' in the occurrence of losses, i.e. a highly bursty loss process. As can be seen, the CLP is indeed very high and more or less independent of the buffer capacity N. Nevertheless, a frequently used method of estimating the loss characteristics in a traffic stream is the *independence* assumption, i.e. to assume that every loss occurs independently from each other. Under this assumption, we have that CLP = PLR, which is a severe underestimation as can be seen from Fig. 9. In other words, under the independence assumption the losses are well spread over time, while in reality they are much more clustered. On the other hand, the probability $1 - \text{CAP}$ that a packet is lost given that the previous one was *accepted* can be approximated reasonably well by the PLR (i.e. using the independence assumption) as can be seen from Fig. 10.

4 Conclusions

We have investigated the characteristics of the loss and non-loss periods of an $M/G/1$ queueing system with finite capacity N. Besides the moments of the lengths of these periods and the number of arrivals during these periods, we also established expressions for various other performance measures: the packet loss ratio, the conditional loss probability and the conditional acceptance probability. The analysis relies on the properties developed in the Appendix, which are interesting in their own right. These properties basically use the well-known PASTA property to relate (parts of) the system state at random time points to the system state as observed upon a random arrival under specific conditions. In addition, we have illustrated our results by means of some relevant numerical examples with various transmission time distributions.

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Appendix

Consider a renewal process and let $S(\zeta)$ and $\mathrm{E}[S]$ denote the common Laplace-Stieltjes transform and the mean of the consecutive renewal periods respectively. On the same time axis, we further consider a Poisson arrival process with intensity λ. We now retrieve expressions for various number- and time-related characteristics of these processes. With respect to the application of these properties to the analysis in section 1 and 2, we note that the renewal periods here correspond to the independent transmission times of the packets. Also, it is important to remark that the arrival process of the $M/G/1/N$ queueing system confined to the busy periods of the $M/G/1/N$ is *still* a Poisson process with the same intensity λ.

Property 1. *Consider a random (tagged) arrival and let T_B and T_A denote the amount of time since the last renewal epoch and until the next renewal epoch respectively. We find,*

$$\Upsilon(\zeta_B,\zeta_A) = \mathrm{E}\left[e^{-\zeta_B T_B - \zeta_A T_A}\right] = \frac{S(\zeta_B) - S(\zeta_A)}{\mathrm{E}[S](\zeta_A - \zeta_B)}\,, \tag{14}$$

$$\Upsilon(\zeta_B,\zeta_A,z_B,z_A) = \mathrm{E}\left[e^{-\zeta_B T_B - \zeta_A T_A} z_B^{A_B} z_A^{A_A}\right] = \frac{S(\zeta_B+\lambda(1-z_B)) - S(\zeta_A+\lambda(1-z_A))}{\mathrm{E}[S](\zeta_A - \zeta_B + \lambda z_A - \lambda z_B)}\,. \tag{15}$$

Proof. Equation (14) immediately follows from PASTA (Wolff 1982) and the renewal theorem (Takagi 1991, p. 18). Further, the number of arrivals before and after the tagged arrival are Poisson distributed random variables with mean λT_B and λT_A respectively. This observation then yields equation (15). □

Property 2. *Consider a random renewal period and let A and S denote the number of arrivals during this renewal period and the length of this renewal period respectively. We then find the following expressions,*

$$\Theta(\zeta,z) = S(\zeta+\lambda(1-z))\,, \tag{16}$$

with $\mathcal{C}$ a contour around the origin within the unit disk.

Proof. The number of arrivals during a renewal period with length S is Poisson distributed with mean λS which immediately leads to (16). $\square$

Property 3. *Consider a random renewal period and let A denote the number of arrivals during this renewal period. Further, let L denote the time between the last arrival during the renewal period if there are arrivals ($A > 0$) or equal the length of the renewal period if this is not the case. We then find,*

$$\Psi(\zeta, z) = \mathrm{E}\left[e^{-\zeta L} z^A\right] = \frac{\lambda z S(\lambda(1-z)) + \zeta S(\zeta+\lambda)}{\zeta + \lambda z}\,, \tag{17}$$

with $\mathcal{C}$ a contour around the origin within the unit disk.

Proof. Consider a random renewal period and let S denote the length of this renewal period. Conditioning on the number of arrivals A during this renewal period then yields,

$$\Psi(\zeta, z) = \mathrm{E}\left[e^{-\zeta L} z^A | A > 0\right](1 - S(\lambda)) + S(\lambda + \zeta)\,. \tag{18}$$

Here we used Property 2 and the fact that $L = S$ for $A = 0$. Clearly, there is a one-to-one correspondence between renewal periods with at least one arrival and arrivals that are not followed by any other arrivals during the same renewal period, say last arrivals. Further, a random arrival is a last arrival if there are no more arrivals during the same renewal period after this arrival. Therefore we find,

$$\mathrm{E}\left[e^{-\zeta L} z^A | A > 0\right] = \mathrm{E}\left[e^{-\zeta T_A} z^{A_B+1} | A_A = 0\right]\,, \tag{19}$$

where T_A, A_A and A_B are defined as in Property 1. By means of some standard transform manipulations, the right-hand side of the former equation can be expressed in terms of Υ as given in Property 1,

$$\mathrm{E}\left[e^{-\zeta T_A} z^{A_B+1} | A_A = 0\right] = z\frac{\Upsilon(0, \zeta, z, 0)}{\Upsilon(0, 0, 1, 0)}\,. \tag{20}$$

Combining equations (18) to (20) and Property 1 then yields (17). $\square$

Property 4. *Consider (tag) a random arrival and let T_B and A_B denote the time since the beginning of the renewal period and the number of arrivals during the renewal period but before the tagged arrival respectively. We have,*

$$\Omega(\zeta; n) = \mathrm{E}\left[e^{-\zeta T_B} 1(A_B = n)\right] = \frac{1}{2\pi\imath}\oint_{\mathcal{C}} \frac{1 - S(\zeta + \lambda(1-y))}{\mathrm{E}[S](\zeta + \lambda(1-y))} y^{-n-1} dy\,, \tag{21}$$

for $n = 0, 1, 2, \ldots$, and with $\mathcal{C}$ a contour around the origin within the unit disk.

Proof. The result (21) immediately follows by inverse z-transforming $\Upsilon(\zeta, 0, 1, z)$ as given in Property 1. $\square$

Property 5. *Consider (tag) a random arrival and let A_B denote the number of arrivals during the tagged arrival's renewal period but before the tagged arrival. Further, let $\tilde{L}$ denote the time between the tagged arrival and the preceding arrival if there is such an arrival during the renewal period ($A_B > 0$) or equal the time since the start of the renewal period if this is not the case ($A_B = 0$) . We then find,*

$$\Gamma(\zeta, z) = \mathrm{E}\left[e^{-\zeta \tilde{L}} z^{A_B}\right] = \frac{\zeta + \lambda z - z(\zeta+\lambda)S(\lambda(1-z))}{\mathrm{E}[S](\zeta+\lambda)(1-z)(\zeta+\lambda z)} - \frac{\zeta\, S(\zeta+\lambda)}{\mathrm{E}[S](\zeta+\lambda)(\zeta+\lambda z)}\,, \tag{22}$$

with $\mathcal{C}$ a contour around the origin within the unit disk.

Proof. Consider a series of random arrivals. Since random arrivals occur during different renewal periods almost surely, the periods of time between the beginning of the renewal period and the arrival instant of the random arrivals constitute a series of independent and identically distributed random variables. I.e., confining time to these periods yields a new renewal process. By means of Property 1, one sees that their common LST equals $\Upsilon(\zeta, 0)$. Also, the arrival process during confined time is a Poisson process with the same arrival rate λ.

In confined time, one identifies the last arrival before a random arrival with the last arrival during a renewal period of the (modified) renewal process. Therefore, application of Property 3 on the modified renewal process yields the stated result (22). $\square$

Property 6. *Consider (tag) a random renewal period and let A denote the number of arrivals during this renewal period. Further, let $\overline{L}_n$ denote the time between the nth arrival and the last arrival during the tagged renewal period if there are at least n arrivals or equal 0 if this is not the case. We then find,*

$$\begin{aligned}(23)\qquad \Xi(\zeta,z;n) &= \mathrm{E}\left[e^{-\zeta\overline{L}_n}z^A\right]\\ &= \frac{1}{2\pi\imath}\oint_{\mathcal{C}}\frac{S(\lambda(1-y))}{z-y}\left(\frac{\zeta(z/y)^{n+1}}{\zeta+\lambda(y-z)}-1\right)dy+\left(\frac{\lambda z}{\lambda z-\zeta}\right)^{n+1}S(\zeta+\lambda(1-z))\,,\end{aligned}$$

for $n=1,2,\ldots,$ and with $\mathcal{C}$ a contour around the origin within the unit disk.

Proof. Since for $A\leq n$ we have $\overline{L}_n=0$, conditioning on A yields,

$$(24)\qquad \Xi(\zeta,z;n)=\vartheta_n(z)+(1-\vartheta_n(1))\,\mathrm{E}\left[e^{-\zeta\overline{L}_n}z^A|A>n\right].$$

where $\vartheta_n(z)$ is given by,

$$\begin{aligned}(25)\qquad \vartheta_n(z) &= \mathrm{E}\left[z^A 1(A\leq n)\right]\\ &= \frac{1}{2\pi\imath}\sum_{m=0}^{n}z^m\oint_{\mathcal{C}}S(\lambda(1-y))y^{-m-1}dy\\ &= \frac{1}{2\pi\imath}\oint_{\mathcal{C}}S(\lambda(1-y))\frac{(z/y)^{n+1}-1}{z-y}\,dy\,.\end{aligned}$$

Let us now confine time to the periods after the nth arrival in renewal periods where there are at least n arrivals. Clearly, the lengths of these consecutive periods constitute a series of independent and identically distributed random variables. In view of Property 1 and by means of inverse z-transformation, their common LST is given by,

$$(26)\qquad \chi(\zeta;n)=\mathrm{E}\left[e^{-\zeta T_A}|A_B=n-1\right]=\frac{\oint_{\mathcal{C}}\Upsilon(\lambda(1-y),\zeta)y^{-n}dy}{\oint_{\mathcal{C}}\Upsilon(\lambda(1-y),0)y^{-n}dy}\,.$$

Further, the restriction of the arrival process to confined time is still a Poisson process.

Consider now a random (tagged) arrival in (confined) time and let $\hat{T}_B$ and $\hat{T}_A$ denote the time between the preceding renewal epoch and the tagged point and the time between this point and the next renewal epoch respectively. In view of Property 1, the joint LST of $\hat{T}_B$ and $\hat{T}_A$ is given by,

$$(27)\qquad \hat{\Upsilon}_n(\zeta_B,\zeta_A)=\frac{\chi(\zeta_B;n)-\chi(\zeta_A;n)}{-\chi'(0;n)(\zeta_A-\zeta_B)}\,.$$

In terms of the original renewal process, $\hat{\Upsilon}_n$ is the joint transform of the time between the nth arrival and a random arrival after the nth arrival and between this random arrival and the end of the renewal period in a renewal period where there are more than n arrivals. Since the last arrival during a renewal period is a random arrival that is not followed by other arrivals during the same renewal period, we have,

$$(28)\qquad \mathrm{E}\left[e^{-\zeta\overline{L}_n}|A>n\right]=\frac{\hat{\Upsilon}_n(\zeta,\lambda)}{\hat{\Upsilon}_n(0,\lambda)}\,.$$

Finally, the total number of arrivals during the tagged renewal period equals the number of arrivals before the nth arrival, the nth arrival, the number of arrivals between the nth and the last arrival and the last arrival, we have,

$$(29)\qquad \mathrm{E}\left[e^{-\zeta\overline{L}_n}z^A|A>n\right]=\frac{\hat{\Upsilon}_n(\zeta+\lambda(1-z),\lambda)}{\hat{\Upsilon}_n(0,\lambda)}z^{n+1}\,.$$

Combining equations (24) to (29) yields equation (23). □